MULTIGRID TECHNIQUE WITH LOCAL GRID REFINEMENT FOR SOLVING STATIC FIELD PROBLEMS

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ABSTRACT

A multilevel iteration technique has been developed for solving the Laplace equation of 2-dimensional static field problems in arbitrary layered structures for the RF circuit design. Multigrid methods are well accepted in the fields of applied mathematics, but are poorly disseminated in numerical applications of electro-magnetic fields. That's why the authors wish to give an introduction to this theory and emphasize the convergence acceleration. The method is verified with measurements of a coplanar capacity line and is compared with conventional solvers.

INTRODUCTION

Multigrid techniques are well known numerical iteration methods to solve partial differential equations (PDE) of any kind. They are well established in solving problems in the fields of applied mathematics like elasticity, deformation, velocity flux, sonic flow, heat transfer, convection diffusion, eigenvalue problems and so on [1,2,3]. Although, the number and complexity of applications and users is growing, the acceptance to determine electro-magnetic fields is very poor. First, this paper will give a short insight into the method and will then demonstrate the improved convergence rate of the multigrid iteration in comparison to conventional solvers. As an enhancement of the method, a local grid refinement (LGR) will be introduced and examined on a coplanar capacity line.

MULTIGRID BASICS

The PDE which will be considered is the Poisson's equation (2nd order, elliptic PDE). The 2-dimensional formulation and its representation in discreet form are

$$\Delta u(x,y) = f(x,y)\,,\ \mathbf{L}^h\mathbf{u}^h = \mathbf{f}^h \text{ in } \Omega^h\,, \qquad \text{(1a,b)}$$

where Ω^h is any domain with the discretization level *h*. $\mathbf{L}^h$ is an operator in form of a sparse matrix and $\mathbf{u}^h$ is the numerical solution of the problem. Conventional iteration procedures to solve (1b) are the Gauss-Seidel, the successive overrelaxation or the Jacobi method. From the literature it is known [2], that these relaxation methods are capable to smoothen the high frequency modes of the error very fast, but not the low-frequency modes. They can be better smoothened on coarser grids. The basic idea of a multigrid scheme now is to smooth the error on a fine grid, calculate the residual, transfer it to the next coarser level where the solution of the error is determined. The step from a fine to a coarse level is called restriction. The prolongation is to interpolate the error from the coarse level back to the fine level and correct the original equation by the error. After this coarse grid correction, a post smoothing should be performed. For the 2-

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grid algorithm the steps can be summarized in the following equations:

1. $\mathbf{L}^h\mathbf{u}^h = \mathbf{f}^h$ — discreet differential equation in level *h*
2. $\tilde{\mathbf{u}}^h = \mathrm{S}^{\mathrm{pre}}(\mathbf{u}^h, \mathbf{f}^h)$ — pre-smoothing $\mathrm{S}^{\mathrm{pre}}$; $\tilde{\mathbf{u}}^h$ is approximation of $\mathbf{u}^h$
3. $\mathbf{r}^h = \mathbf{L}^h\tilde{\mathbf{u}}^h - \mathbf{f}^h$ — residual equation; $\mathbf{r}^h$ is the residual
4. $\mathbf{L}^{2h}\mathbf{e}^{2h} = \mathrm{R}_{2h}^{h}(\mathbf{r}^h)$ — restriction from fine to coarse level R_{2h}^{h}; $\mathbf{e}^h = \tilde{\mathbf{u}}^h - \mathbf{u}^h$
5. $\mathbf{e}^{2h} = \mathrm{S}\left(\mathbf{e}^{2h}, \mathbf{R}_{2h}^{h}\left(\mathbf{r}^h\right)\right)$ — exact solution of the error in the coarsest level (2)
6. $\mathbf{e}^h = \mathrm{I}_h^{2h}(\mathbf{e}^{2h})$ — interpolation from coarse to fine level I_h^{2h}
7. $\mathbf{u}^h = \tilde{\mathbf{u}}^h - \mathbf{e}^h$ — correction
8. $\mathbf{u}^h = \mathrm{S}^{\mathrm{post}}(\mathbf{u}^h, \mathbf{f}^h)$ — post-smoothing $\mathrm{S}^{\mathrm{post}}$

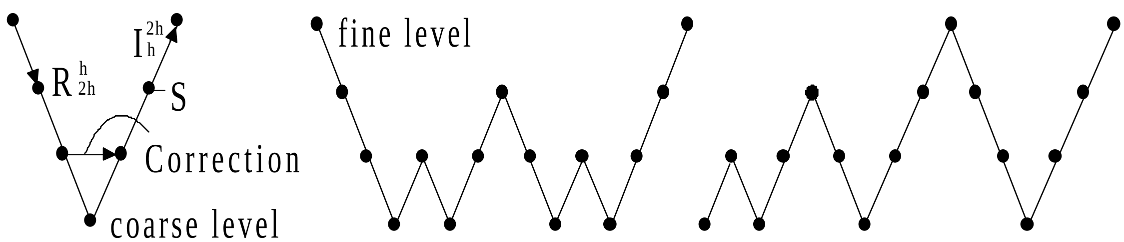


Figure 1: V-, W- and FMG-cycles

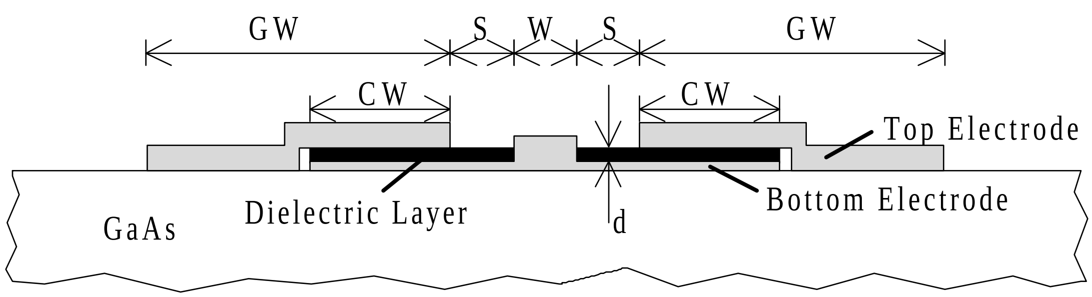


Figure 2: Cross-view of the coplanar capacity waveguide (W = 10 µm, S = 10 µm, CW = 65 µm, GW = 200 µm)

This can be extended to more levels of refinement. Different procedures of level interchanges are possible. Typical cycling techniques are V-, W- or FMG-cycles. FMG stands for full-multi grid. Here, the iteration starts in the coarsest level. Schematic presentations of the different cyclings are shown in figure 1. In this paper, the goal is to determine the static potential of arbitrary 2-dimensional TEM-waveguides. The PDE, which has to be solved with different parameters for the multigrid method, is the Laplace equation, a special form of the Poisson's equation. One difficulty is to find a set of grids with different refinement. The solution is to draw one x- and one y-grid line through each edge of the structure under consideration [6]. The resulting grid will be taken as the coarsest level of refinement. In each finer level, a new grid line is placed in the middle of two lines of the coarser level. Such a bisection will not be performed, if a minimal discretization distance has been reached. This factor for the x- and y-direction as a function of the discretization can be determined automatically by the software. The result is an almost homogeneous grid in the finest level [5]. Nevertheless, field-singularities at the edges of conductors affect a slow down of the convergence of the iteration. That's why a local refinement technique, described in [4] is used to accelerate the iteration measured as the asymptotic convergence rate (ACR). The ACR is defined by $\lim_{i\to\infty}(R_{i+1}/R_i)$, where R_i is any fixed residual norm of $\mathbf{r}^h$, measured after the *i*th cycle. The principle is to perform a few relaxation sweeps around the grid points with singularities before the post-smoothing on the regular grid is done. Some iteration techniques are compared in the diagram of figure 3.

As expected, the Gauss-Seidel method has the slowest convergence rate followed by the successive overrelaxation (SOR with a relaxation factor of 1.8). The standard multigrid algorithm with V- or W-cycling already shows an improved convergence. This can be enhanced simply by adding a few relaxation steps on the local refined mesh close to the edges of the structure (see example of figure 2).

EXPERIMENTAL RESULTS

An application, that has been chosen to verify the multigrid solver, is a coplanar waveguide with a high capacitance to ground realized in the MIM technology. This structure is suitable, because the resulting equivalent circuit elements can be compared with the values of a similar parallel plate capacitor as well as with the data of a measured circuit, fabricated on GaAs. The cross view of the structure is shown in figure 2. The element is very applicable in bias supply circuits, since the large size oft the capacitance's area can completely vanish under the ground stripes of the coplanar line. The multigrid solver determines the equivalent line parameters from the static fields to $C' = 47.96\,\frac{\text{nF}}{\text{m}}$ and the capacitance in units per length for the structure filled with air instead of dielectric materials to $C_0' = 6.71\,\frac{\text{nF}}{\text{m}}$. This means, the equivalent inductance $L_0' = (\mu_0\varepsilon_0)/C_0' = 1.66\,\frac{\text{nH}}{\text{m}}$, if the structure can be regarded as a TEM waveguide. The differences to a parallel plate capacitor are the scattered fields in the center and at the edges of the coplanar line. Since the length and the capacitor width (l=500µm and CW=65µm) of the investigated structure are large, the equivalent line parameters should be close to those of the parallel plate capacitor. The numbers are: $C_{\|}' = \dfrac{\varepsilon_0\varepsilon_r 2CW}{d} = 45.48\,\frac{nF}{m}$ and $C_{0\|}' = 6.22\,\frac{\text{nF}}{\text{m}}$.

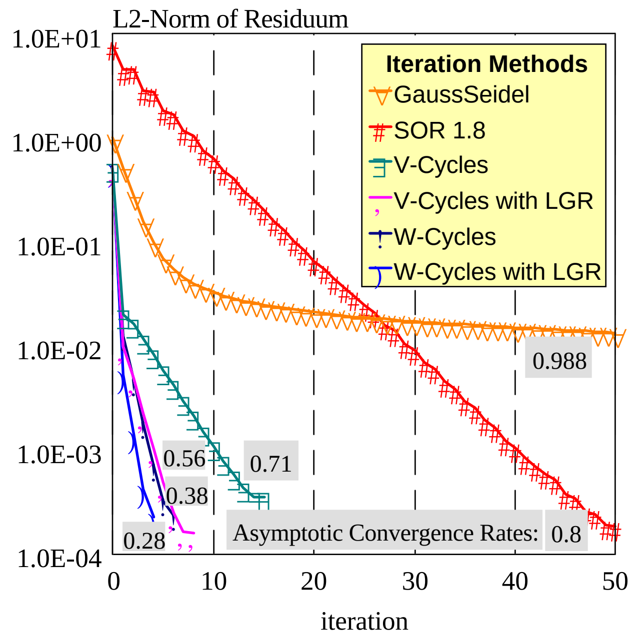
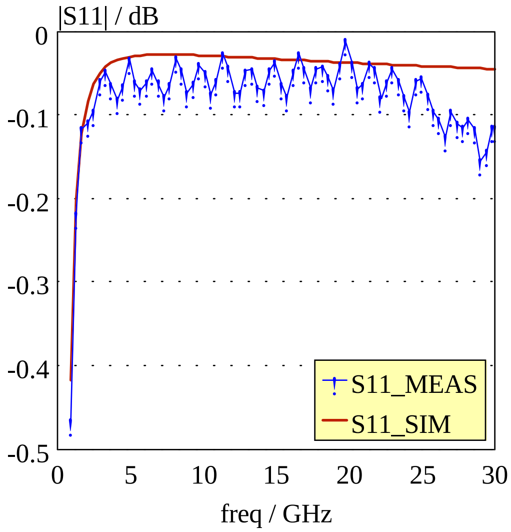


Figure 3: Convergence behavior of various iteration methods

The coplanar capacity line has been fabricated on GaAs at the Daimler Benz research center in Germany (Ulm). The line length is 500µm. Figure 4 shows a comparison of measured and simulated results up to 30 GHz. The agreement is outstanding. With the combination of an accurate modeling tool and very short simulation times this software is utmost suitable for computer aided circuit design tools.



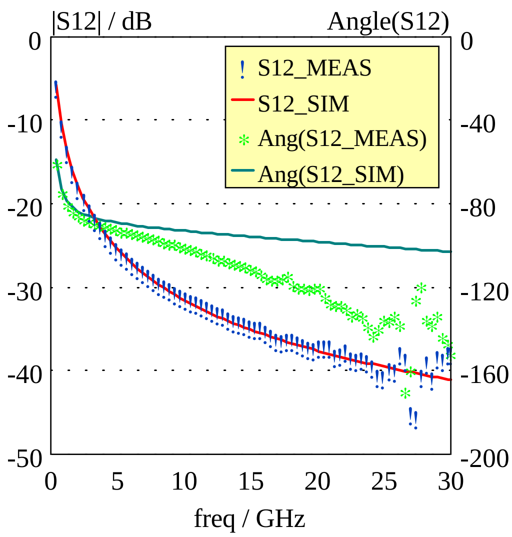


Figure 4: Measured and simulated S-parameters of the coplanar capacity line

CONCLUSION

An iteration method for solving static field problems has been presented, which is based on multigrid / multilevel methods with local grid refinements. It has been shown, that an acceleration compared to the conventional Gauss-Seidel method can be achieved. This method with its automatic mesh generator is well suited for the use in CAD tools.

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